

REGULARITY OF SCHRAMM-LOEWNER EVOLUTIONS, ANNULAR CROSSINGS, AND ROUGH PATH THEORY

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ABSTRACT. When studying stochastic processes, it is often fruitful to understand several different notions of regularity. One such notion is the optimal Hölder exponent obtainable under reparametrization. In this paper, we show that chordal SLE_κ in the unit disk for $\kappa \leq 4$ can be reparametrized to be Hölder continuous of any order up to $1/(1 + \kappa/8)$.

From this, we obtain that the Young integral is well defined along such SLE_κ paths with probability one, and hence that SLE_κ admits a path-wise notion of integration. This allows us to consider the expected signature of SLE, as defined in rough path theory, and to give a precise formula for its first three gradings.

The main technical result required is a uniform bound on the probability that an SLE_κ crosses an annulus k -distinct times.

1. INTRODUCTION

Oded Schramm introduced Schramm-Loewener Evolutions (SLE) as a stochastic process to serve as the scaling limit of various discrete models from statistical physics believed to be conformally invariant in the limit [21]. It has successfully been used to study a number of such processes (for example, loop-erased random walk and uniform spanning-tree [10], percolation exploration process [24], Gaussian free field interfaces [22], and Ising model cluster boundaries [25]).

To define SLE, it is convenient to parametrize the curve so that the half-plane capacity increases linearly and deterministically with time – a change which allowed the use of a form of the Loewner differential equation. This has proven an extremely fruitful point of view, enabling the definition of SLE_κ and the proof of all of the above convergence results.

However, in doing so, the original parametrizations of the discrete models are lost along with any information about the regularity of these parameterizations. To try and recover some information on the possible regularity properties of these parametrizations, it is reasonable to ask the question: what are

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the best regularity properties that SLE_κ curves can have under any *arbitrary* reparametrization?

Regularity of SLE_κ under the capacity parametrization is well understood. In [7], Johansson Viklund and Lawler prove a conjecture of Lind from [15] that for chordal SLE_κ parametrized by capacity, the optimal Hölder exponent is

$$\alpha_0 = \min \left\{ \frac{1}{2}, 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8 + \kappa}} \right\}.$$

However, this value differs greatly from what one might expect. In [2], Beffara shows that the almost sure Hausdorff dimension of a chordal SLE_κ is $1 + \kappa/8$. A d -dimensional curve γ cannot be reparametrized to be Hölder continuous of any order greater than $1/d$, and intuition from other stochastic processes implies that SLE_κ should be able to be reparametrized to be Hölder continuous of all remaining orders, which is not what we see under capacity parametrization.

In this paper, we answer this question for $\kappa \leq 4$ and show that the best possible result is true.

Theorem 1.1. *Fix $0 \leq \kappa \leq 4$ and let $\gamma : [0, \infty] \rightarrow \mathbb{D}$ be a chordal SLE_κ from 1 to -1 in \mathbb{D} and $d = 1 + \kappa/8$ be its almost sure Hausdorff dimension. Then, with probability one the following holds:*

- *for any $\alpha < 1/d$, γ can be reparametrized as a curve $\tilde{\gamma} : [0, 1] \rightarrow D$ which is be Hölder continuous of order α , and*
- *for any $\alpha > 1/d$, γ cannot be reparametrized as a curve $\tilde{\gamma} : [0, 1] \rightarrow D$ which is be Hölder continuous of order α .*

The critical case of $\alpha = 1/d$ is still open, however it is natural to conjecture that it cannot be reparametrized to be Hölder continuous of this order.

With this result, we are able to provide a few preliminary results in the rough path theory of SLE.

First, we obtain a definition integration against a SLE_κ curve. In particular, this result shows that SLE_κ for $\kappa \leq 4$ has finite d -variation for some $d < 2$ in the sense used in [18] and thus both the Young integral and the integral of Lions as defined in [16] give a way of almost surely integrating path-wise along an SLE curve, and moreover iterating such integrals to provide almost sure existence of differential equations driven by an SLE curve.

Second, we provide a partial computation of the expected signature for SLE_κ in the disk. In rough path theory, the expected signature is a non-commutative power series which is regarded as a kind of non-commutative Laplace transform for paths. It is believed to characterize the measure on the path up to an appropriate sense of equivalence of paths [4, 6]. We provide a computation of the first three gradings of this non-commutative power series.

The main technical tool used to prove these results is a result by Aizenman and Burchard in [1] which states, informally, that all that is needed to obtain a certain degree of Hölder regularity in a random curve is a uniform estimate on the probabilities that the curve crosses an annulus k distinct times. In particular, we obtain the following result for SLE_κ in \mathbb{D} . Let $A_r^R(z_0)$ denote the annulus with inner radius r and outer radius R centered at z_0 .

Theorem 1.2. *Fix $\kappa \leq 4$. For any $k \geq 1$, there exists c_k so that for any $z_0 \in \mathbb{D}$, $r < R$,*

$$\mathbb{P}\{\gamma \text{ traverses } A_r^R(z_0) \text{ at least } k \text{ separate times}\} \leq c_k \left(\frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)}.$$

Finally, there has been recent work defining a parametrization of SLE, called the *natural parametrization*, which should be the scaling limit for the parametrizations of the discrete curves (see [11, 13] for the theory in deterministic geometries, and [3] for a version in random geometries). It is conjectured that SLE_κ under the natural parametrization should have the optimal Hölder exponent our result indicates, however our techniques do not immediately illuminate this question.

The paper is organized as follows. In Section 2, we review the basic definition of SLE_κ and introduce the notation used throughout. Then, in Section 3, a brief overview of the regularity results needed from [1] is given along with a discussion of their application to SLE_κ . Before proving the regularity result, we present our applications by providing a definition of integration against an SLE path and the computation of the first three gradings of the expected signature in Section 4. Finally, in Section 5, we prove the main estimate bounding the probability that an SLE_κ crosses an annulus at least k times.

We consider $0 \leq \kappa \leq 4$ to be fixed and write $a := 2/\kappa$. All constants throughout may implicitly depend on κ .

2. SLE DEFINITION AND NOTATION

We first review the definition of chordal SLE_κ in \mathbb{H} . For a complete introduction to the subject, see, for example, [8, 9, 26]. For any $\kappa \geq 0$, let $a = 2/\kappa$ and define $g_t(z)$ to be the unique solution to

$$\partial_t g_t(z) = \frac{a}{g_t(z) + B_t}, \quad g_0(z) = z.$$

where B_t is a standard Brownian motion. We refer to this equation as the *chordal Loewner equation* in \mathbb{H} , and the Brownian motion is the *driving function*.

For any $z \in \mathbb{H}$ this is well defined up to some random time T_z . Let $H_t = \{z \in \mathbb{H} \mid T_z > t\}$ be the set of points for which the solution is well defined up until time t . The chordal Loewner equation is defined so that $g_t : H_t \rightarrow \mathbb{H}$

is the unique conformal map from H_t to \mathbb{H} which fixes infinity with $g_t(z) = z + \frac{at}{z} + O(z^{-2})$ as $z \rightarrow \infty$.

It was shown by Rohde and Schramm in [20] that for any value of $\kappa \neq 8$ there exists a unique continuous curve $\gamma : [0, \infty) \rightarrow \overline{\mathbb{H}}$ such that $H_t = \mathbb{H} \setminus \gamma[0, t]$. This holds for $\kappa = 8$ as well, however the proof in this case differs significantly [10]. This curve is *chordal SLE $_{\kappa}$ from 0 to ∞ in \mathbb{H}* .

To define SLE $_{\kappa}$ in other simply connected domains D from $z_1 \in \partial D$ to $z_2 \in \partial D$, let $f : \mathbb{H} \rightarrow D$ be a conformal map so that $f(0) = z_1$ and $f(\infty) = z_2$, and define SLE $_{\kappa}$ in this new domain by taking the image of the curve γ under this conformal map.

In this work, we mainly consider SLE $_{\kappa}$ from 1 to -1 in \mathbb{D} . In this case, it is known that $\gamma(\infty^-) = -1$ (see, for example, [8, Chapter 6] for the proof in \mathbb{H}), and hence we extend γ to be well defined on the times $[0, \infty]$. While this particular choice of domain and boundary points is not required for the work that follows, it is important to choose a domain with sufficiently smooth boundary to avoid detrimental boundary effects (for instance, domains and boundary points without any Hölder continuous curves between them).

3. TORTUOSITY, HÖLDER CONTINUITY, AND DIMENSION

3.1. Definitions and relations. To prove the claimed order of continuity, we need to use tools first described by Aizenman and Burchard in [1]. This review is devoid of proofs, which the interested reader may find in the original paper along with many results beyond what are needed in this paper. To aid in this, we have included the original theorem numbers for each result with each statement. We begin by describing three different measures of regularity and their deterministic relationships. Throughout this section $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ is a compact continuous curve in \mathbb{R}^d .

First, recall that a curve $\gamma(t)$ is *Hölder continuous of order α* if there exists a constant C_α such that for all $0 \leq s \leq t \leq 1$, we have that $|\gamma(s) - \gamma(t)| \leq C_\alpha |t - s|^\alpha$. This condition becomes stricter for larger values of α , thus if one wants to turn this into a parametrization independent notion of regularity it makes sense to define

$$\alpha(\gamma) = \sup\{\alpha \mid \gamma \text{ admits an } \alpha\text{-Hölder continuous reparametrization}\}.$$

While a familiar and useful notion of regularity, it can be hard to work with directly for random curves. A similar notion, which is more amenable to estimation is the concept of the *tortuosity*. Let $M(\gamma, \ell)$ denote the minimal number of segments needed to partition the curve γ into segments of diameter no greater than ℓ . As with most of these dimension like quantities, we wish to understand its power law rate of growth, thus we define the *tortuosity exponent* to be

$$\tau(\gamma) = \inf\{s > 0 \mid \ell^s M(\gamma, \ell) \rightarrow 0 \text{ as } \ell \rightarrow 0\}.$$

These two notions are similar in so far as they define a type of regularity for a curve in a local way which is, to a large extent, insensitive to the large scale geometry of the curve. As such, one should not be surprised that they are deterministically related by the following result.

Theorem 3.1 ([1, Theorem 2.3]). *For any curve $\gamma : [0, 1] \rightarrow \mathbb{R}^d$,*

$$\tau(\gamma) = \alpha(\gamma)^{-1}.$$

Often times, it is easier still to estimate a quantity which takes global geometry in to account, in particular we discuss the *upper box dimension*. Let $N(\gamma, \ell)$ denote the minimal number of sets of diameter at most ℓ needed to cover γ . Then the upper box dimension is

$$\dim_B(\gamma) = \inf\{s > 0 \mid \ell^s N(\gamma, \ell) \rightarrow 0 \text{ as } \ell \rightarrow 0\}.$$

The upper box dimension can differ quite markedly from the tortuosity exponent as a single set in the cover can contain a large number of different segments of γ of similar diameter. In fact, there exist curves in the plane which cannot be parametrized to be Hölder continuous of any order, and hence $\tau(\gamma) = \infty$, while $\dim_B(\gamma) \leq d$ for any compact curve $\gamma : [0, 1] \rightarrow \mathbb{R}^d$. In general, it is immediate from the definitions that $\dim_B(\gamma) \leq \tau(\gamma)$, however the inequality can be strict.

What is desired is a condition which deterministically ensures that the upper box dimension and the tortuosity exponent coincide, allowing us to control the optimal Hölder exponent with the upper box dimension.

Aizenman and Burchard provide such a property which they refer to as the *tempered crossing property*. Say that a curve γ exhibits a k -fold crossing of power ε at the scale $r \leq 1$ if it traverses some spherical shell of the form

$$D(x; r^{1+\varepsilon}, r) := \{y \in \mathbb{R}^d \mid r^{1+\varepsilon} \leq |y - x| \leq r\}.$$

With this, we say a curve has the tempered crossing property if for every $0 < \varepsilon < 1$ there exists $k(\varepsilon)$ and $0 < r_0(\varepsilon) < 1$ such that on scales smaller than $r_0(\varepsilon)$, the curve has no $k(\varepsilon)$ -fold crossings of power ε .

Theorem 3.2 ([1, Theorem 2.5]). *If $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ has the tempered crossing property,*

$$\tau(\gamma) = \dim_B(\gamma).$$

Thus, the goal is to find a simple probabilistic condition which ensures that with probability one the tempered crossing property holds. We present a weaker form of the theorem than is found in [1].

Theorem 3.3 ([1, Lemma 3.1]). *Let $\gamma : [0, 1] \rightarrow \Lambda$ be a random curve contained in some compact set $\Lambda \subseteq \mathbb{R}^d$. If for all k there exists c_k and $\lambda(k)$ so*

that for all $x \in \Lambda$ and all $0 < r \leq R \leq 1$ we have

$$\mathbb{P}\{\gamma \text{ traverses } D(x; r, R) \text{ at least } k \text{ separate times}\} \leq c_k \left(\frac{r}{R}\right)^{\lambda(k)}$$

where additionally $\lambda(k) \rightarrow \infty$ as $k \rightarrow \infty$, then the tempered crossing probability holds almost surely, and hence

$$\dim_B(\gamma) = \tau(\gamma) = \alpha(\gamma)^{-1}.$$

3.2. SLE specific bounds. We need two ingredients to apply the techniques of the previous section to SLE_κ and obtain Theorem 1.1.

First, we need to prove that SLE_κ from 1 to -1 in \mathbb{D} satisfies the conditions of Theorem 3.3. This is the main work of this paper and the result, Theorem 1.2, is proven in Section 5. This estimate shows that for any $\kappa \leq 4$ the tempered crossing property holds with probability one, and hence $\alpha(\gamma)^{-1} = \dim_B(\gamma)$. Note that the condition that the curve has a finite parametrization is immaterial since we may turn the normally infinite parametrization of an SLE_κ curve to a finite one by precomposing by an appropriate function.

Second, we need to know the upper box dimension is $1 + \kappa/8$ with probability one. This is a consequence of a pair of well known results. From [20, Theorem 8.1] we obtain that the upper box dimension is bounded above by the desired value, while the lower bound can be obtained by noting the Hausdorff dimension is $1 + \kappa/8$ almost surely (proven by Beffara in [2]) and using that the Hausdorff dimension is a lower bound for the box dimension.

4. INTEGRALS AND ROUGH PATH THEORY

Before proving our regularity result, we discuss a few applications to integration along SLE paths and the rough path theory of SLE.

4.1. d -variation and integrals. With the main regularity result, we may prove the existence of integrals of the form

$$\int_0^t f(s) \, d\gamma(s)$$

when γ is an SLE_κ and f is a sufficiently nice function. In particular, both the Young integral (as first defined in [27], and used in rough path theory) and the integral defined by Lions in [16] are well defined with probability one for SLE_κ with $\kappa \leq 4$. For simplicity we discuss only the Young integral – checking the condition Lions' integral is similar.

Given a continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^d$, let $\|\gamma\|_p$ denote the p -variation of γ , defined as

$$\|\gamma\|_p = \left[\sup_{\mathcal{P}} \sum_{i=1}^{\#\mathcal{P}} |\gamma(t_i) - \gamma(t_{i-1})|^p \right]^{1/p}$$

where the supremum is taken over partitions $\mathcal{P} = \{t_0, \dots, t_n\}$ of $[0, 1]$. This notion of p -variation is not the one most commonly used elsewhere in probability which would have $\limsup_{|\mathcal{P}| \rightarrow 0}$ in place of the supremum where $|\mathcal{P}|$ is the mesh of the partition. Let $\mathcal{V}^p(\mathbb{R}^d)$ denote the set of all $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ with finite p -variation.

It is immediate from the definitions that if $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ is Hölder continuous of order $1/p$, then it is an element of $\mathcal{V}^p(\mathbb{R}^d)$. Thus for SLE_κ in \mathbb{D} from 1 to -1 , the main regularity result implies that a sample path γ has finite p variation for all $p > 1 + \kappa/8$ with probability one.

The following theorem contains the definition of the Young integral (for a proof see, for example, [18, Theorem 1.16]).

Theorem 4.1. *Fix $p, q > 0$ such that $1/p + 1/q > 1$. Take $f \in \mathcal{V}^q(\mathbb{R})$, and $g \in \mathcal{V}^p(\mathbb{R}^d)$, then for every $t \in [0, 1]$, we have*

$$\int_0^t f(s) \, dg(s) := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{j=1}^{\#\mathcal{P}} f(t_j)(g(t_j) - g(t_{j-1})),$$

where \mathcal{P} is a partition of $[0, t]$, exists and is called the Young integral. Moreover, when considered as a function of t , the Young integral is an element of $\mathcal{V}^p(\mathbb{R}^d)$ and the integral depends continuously on f and g under their respective norms.

Thus, we may integrate any element of $\mathcal{V}^q(\mathbb{R})$ for some $q < (8 + \kappa)/\kappa$ against an SLE_κ sample path with probability one. This includes functions such as Lipschitz functions of γ itself (since $1 + \kappa/8 < (8 + \kappa)/\kappa < 2$ when $\kappa < 8$). Thus, via Picard iteration, we may define ordinary differential equations driven by SLE_κ in a path-wise manner (see [18] for details of the general theory).

When working with these integrals, one often wants to apply results from standard calculus. Luckily, this may frequently be done using the following density result. Given a curve $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ and a partition $\mathcal{P} = \{0 = t_0, t_1, \dots, t_n = 1\}$ of $[0, 1]$, let $\gamma^\mathcal{P}$ be the piecewise linear approximation to γ obtained by linearly interpolating between $\gamma(t_i)$ and $\gamma(t_{i+1})$ for each i .

Proposition 4.2 ([18, Proposition 1.14]). *Let p and q be such that $1 \leq p < q$ and take $\gamma \in \mathcal{V}^p(\mathbb{R}^d)$. Then, $\gamma^\mathcal{P}$ tends to γ in \mathcal{V}^q norm as the mesh of \mathcal{P} tends to zero. Additionally, this convergence may be taken simultaneously in supremum norm.*

Using this approximation technique, statements about integrals against functions in $\mathcal{V}^p(\mathbb{R}^d)$ for $p < 2$ may be reduced to questions about the classical Stieltjes integral, as it and the Young integral are identical for piecewise linear functions. In our case, we want to ensure that we may find an approximating sequence which is simple. The proof of the following lemma is due to Laurence Field [5].

Lemma 4.3. *Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a simple curve. Then for any $\varepsilon > 0$, there exists a partition \mathcal{P} with the mesh of \mathcal{P} less than epsilon and $\gamma^{\mathcal{P}}$ simple.*

Proof. We prove a slightly stronger fact that there exists a partition \mathcal{P} such that for each not only is the mesh of \mathcal{P} smaller than ε , but so is $|\gamma(t_i) - \gamma(t_{i-1})|$ for all i . Since γ is simple, it is a homeomorphic to its image, and thus both γ and γ^{-1} are uniformly continuous. Thus, we may find $\delta_1 < \varepsilon$, $\delta_2 < \varepsilon$, δ_3 , and δ_4 so that

$$\begin{aligned} |t - s| < \delta_1 &\implies |\gamma(t) - \gamma(s)| < \varepsilon, \\ |\gamma(t) - \gamma(s)| < \delta_2 &\implies |t - s| < \delta_1, \\ |t - s| < \delta_3 &\implies |\gamma(t) - \gamma(s)| < \delta_2, \text{ and} \\ |\gamma(t) - \gamma(s)| < \delta_4 &\implies |t - s| < \delta_3. \end{aligned}$$

Let Σ be the set of all times $s \in [0, 1]$ with times $0 = t_0 < t_1 < \dots < t_n = s$ so that

- $t_i - t_{i-1} < \varepsilon$ and $|\gamma(t_i) - \gamma(t_{i-1})| < \varepsilon$ for all i ,
- The curve η obtained by concatenating the segments between $\gamma(t_{i-1})$ and $\gamma(t_i)$ is simple, and
- $\gamma^{-1}(\eta) \subseteq [0, s]$.

We first show that $\sup \Sigma \in \Sigma$. Suppose not, and take times $0 = t_0 < t_1 < \dots < t_n$ as above with $|\gamma(t_n) - \gamma(\sup \Sigma)| < \delta_4$. By shortening the sequence of times t_i , we may assume that $\gamma(t_k)$ is closest to $\gamma(\sup \Sigma)$ when $k = n$. By the choice of δ_4 , we know that $|t_n - \sup \Sigma| < \delta_3$. Let t_{n+1} be the maximum time s so that $\gamma(s)$ is contained in the interval between $\gamma(t_n)$ and $\gamma(\sup \Sigma)$. t_{n+1} is at least $\sup \Sigma$ in size, so we will be done as long as $t_{n+1} \in \Sigma$. The first condition is satisfied since since $|\gamma(t_{n+1}) - \gamma(t_n)| < \delta_2 < \varepsilon$ and thus $t_{n+1} - t_n < \delta_1 < \varepsilon$. The second condition holds since a non-trivial intersection of the final interval with any previous one would force either non-simplicity of γ or a violation of the third condition for the curve up to time t_n . The third condition holds by the definition of t_{n+1} .

We now show that $\sup \Sigma = 1$. Suppose not, and take times $0 = t_0 < t_1 < \dots < t_n = \sup \Sigma$ as above. Take any $t \in (t_n, t_n + \delta_3)$ so that $\gamma(t_n)$ is closer to $\gamma(t)$ than it is to any of the intervals of η not containing $\gamma(t)$. As before, let t_{n+1} be the maximum time s such that $\gamma(s)$ is contained in the interval from $\gamma(t_n)$ and $\gamma(t)$ – a time no smaller than t . Analogously to before, one may readily check that this shows $t_{n+1} \in \Sigma$. ■

4.2. Partial Expected Signature for SLE. Once iterated integrals are defined, we may understand the *signature*, which is the fundamental object of study in rough path theory (See [18] for a more detailed introduction to this

field of study). Of particular interest when dealing with random processes is the expected signature of the path [4, 17].

In this section we provide a computation of the first few gradings of the expected signature for γ , an SLE_κ from 0 to 1 in the disk of radius 1/2 about 1/2, which we denote by D .

Let $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{R}$ denote the real and imaginary components of $\gamma : [0, 1] \rightarrow \mathbb{C}$ respectively and define the *coordinate iterated integrals* as

$$\begin{aligned} \gamma^{k_1 k_2 \dots k_n} &:= \int_{0 < t_1 < t_2 < \dots < t_n < 1} \cdots \int d\gamma_{k_1}(t_1) d\gamma_{k_2}(t_2) \cdots d\gamma_{k_n}(t_n) \\ &= \int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} d\gamma_{k_1}(t_1) d\gamma_{k_2}(t_2) \cdots d\gamma_{k_{n-1}}(t_{n-1}) d\gamma_{k_n}(t_n) \end{aligned}$$

defining $\gamma^\emptyset := 1$. It is convenient to let $\mathbf{k} = k_1 k_2 \dots k_n$ denote the multi-index used above.

An important computational tool when dealing with these iterated integrals is the notion of the *shuffle product*, as defined in the following proposition. We say a permutation σ of $r+s$ elements is a *shuffle* of $1, \dots, r$ and $r+1, \dots, r+s$ if $\sigma(1) < \dots < \sigma(r)$ and $\sigma(r+1) < \dots < \sigma(r+s)$.

Proposition 4.4 ([18, Theorem 2.15]). *Let γ be in $\mathcal{V}^p(\mathbb{R}^d)$ for $p < 2$. Then*

$$\gamma^{k_1 \dots k_r} \cdot \gamma^{k_{r+1} \dots k_{r+s}} = \sum_{\text{shuffles } \sigma} \gamma^{k_{\sigma^{-1}(1)} \dots k_{\sigma^{-1}(r+s)}}.$$

We let $\mathbf{e}_{\mathbf{k}} := \mathbf{e}_{k_1} \otimes \cdots \otimes \mathbf{e}_{k_n}$ denote the basis element for formal series of tensors on the standard basis of \mathbb{R}^2 (viewed as \mathbb{C}). Then the *signature* is defined to be

$$\mathcal{S}(\gamma) = \sum_{\mathbf{k}} \gamma^{\mathbf{k}} \mathbf{e}_{\mathbf{k}}$$

where the sum is taken over all multi-indices \mathbf{k} . The expected signature is thus

$$\mathbb{E}[\mathcal{S}(\gamma)] = \sum_{\mathbf{k}} \mathbb{E}[\gamma^{\mathbf{k}}] \mathbf{e}_{\mathbf{k}}.$$

Computing the expected signature of any process is difficult (see, for example, [17] for the computation of the expected signature of Brownian motion upon exiting a disk, where the solution is found in terms a recursive series of PDE), and thus computing the full expected signature of SLE_κ would be a major undertaking. We provide a computation of the first three gradings.

To do so, we use the probability that a point in D is above the curve γ . This is a well known computation in the SLE literature, and was found by Schramm in [23] for SLE in the upper half plane from 0 to ∞ in terms of a hyper-geometric function. We use a simpler form which can be found in [14].

λ	κ	A_κ
0	4	$\frac{1}{48}$
1	$\frac{8}{3}$	$\frac{1}{48}(6\mathcal{K} - 5)$
2	2	$\frac{1}{4}\log(2) - \frac{1}{6}$
3	$\frac{8}{5}$	$\frac{1}{96}(54\mathcal{K} - 49)$
4	$\frac{4}{3}$	$\frac{2}{3}\log(2) - \frac{11}{24}$
5	$\frac{8}{7}$	$\frac{1}{128}(150\mathcal{K} - 137)$
6	1	$\frac{6}{5}\log(2) - \frac{199}{240}$

TABLE 1. A collection of values of A_κ across a range of integer values of λ . $\mathcal{K} := \sum_{i=1}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.91596\dots$ denotes Catalan's constant.

Let $\lambda = \beta - 1 = 4a - 2 = 8/\kappa - 2$. Then the probability that an SLE_κ from 0 to ∞ in \mathbb{H} passes to the right of a point $r_0 e^{i\theta_0}$ is

$$\phi(\theta_0) := C_\kappa \int_0^{\theta_0} \sin^\lambda(t) dt \quad \text{where} \quad C_\kappa^{-1} := \int_0^\pi \sin^\lambda(t) dt = \frac{\sqrt{\pi} \Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda+2}{2})}.$$

Given a point $x+iy \in D$, let $p(x, y)$ be the probability that an SLE_κ from 0 to 1 in D passes below $x+iy$, which by conformal invariance can be obtained by pre-composing the above expression with the conformal map $z \mapsto iz/(1-z)$, which maps D to \mathbb{H} fixing 0 and sending 1 to ∞ .

Proposition 4.5. *Fix $\kappa \leq 4$ and let*

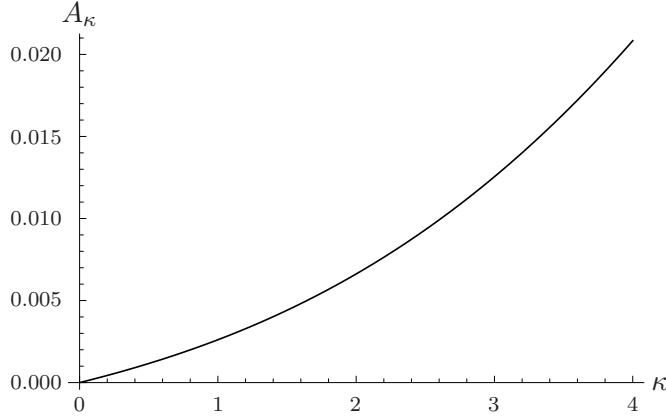
$$\begin{aligned} A_\kappa &= \frac{1}{12} - \int_D y p(x, y) dx dy \\ &= \frac{C_\kappa}{4} \left[\int_0^{\pi/2} \frac{\sin(t) - t \cos(t)}{\sin^3(t)} \cos^\lambda(t) dt \right] - \frac{1}{24}. \end{aligned}$$

Then

$$\mathbb{E}[\mathcal{S}(\gamma)] = 1 + \mathbf{e}_1 + \frac{1}{2}\mathbf{e}_{11} + \frac{1}{6}\mathbf{e}_{111} + A_\kappa \mathbf{e}_{122} - 2A_\kappa \mathbf{e}_{212} + A_\kappa \mathbf{e}_{221} + \dots$$

where γ is an SLE_κ from 0 to 1 in D .

For integer values of λ (which includes the values $\kappa = 4, 8/3$, and 2) this integral can also be evaluated nearly in closed form. Several values of A_κ for integer λ may be found in Table 1. For an understanding of the qualitative behavior of A_κ for other values of κ , we have included a graph in Figure 1.

FIGURE 1. A graph of A_κ as a function of κ .

Proof of Proposition 4.5. First note the iterated integrals defining the signature exist since SLE_κ curves are in $\mathcal{V}^p(\mathbb{C})$ for some $p < 2$.

The initial 1 occurs since γ^\emptyset is defined to be 1. Since every SLE_κ is a curve from 0 to 1,

$$\gamma^1 = \int_0^1 d\gamma_1(t) = \gamma_1(T) - \gamma_1(0) = 1, \quad \gamma^2 = \int_0^1 d\gamma_2(t) = \gamma_2(T) - \gamma_2(0) = 0$$

both with probability one, computing the first grading. Thus, by considering these along with the shuffle products $\gamma^1\gamma^1$, $\gamma^2\gamma^2$, $\gamma^1\gamma^1\gamma^1$, and $\gamma^2\gamma^2\gamma^2$, we get the claimed values for γ^{11} , γ^{22} , γ^{111} , and γ^{222} with probability one, and hence in expectation.

Next, the law of γ is invariant under the map $\gamma \mapsto \bar{\gamma}$. From the definition of the coordinate iterated integral, along with the definition of the Young integral one may see this implies

$$\mathbb{E}[\gamma^{k_1 \dots k_n}] = (-1)^{\#\{i | k_i=2\}} \mathbb{E}[\bar{\gamma}^{k_1 \dots k_n}]$$

and hence any coordinate iterated integral with an odd number of imaginary components in its multi-index must have zero expectation as long as $\mathbb{E}[\gamma^{\mathbf{k}}]$ exists. For the cases we need, one may repeat the Green's theorem argument which follows to show there is some $C_{\mathbf{k}}$ so that $|\gamma^{\mathbf{k}}| \leq C_{\mathbf{k}}$ with probability one and thus conclude immediately that $\mathbb{E}[\gamma^{\mathbf{k}}]$ exists. Thus we have reduced the computation to the terms γ^{122} , γ^{212} , and γ^{221} . By considering the shuffle products

$$0 = \gamma^2 \cdot \gamma^{12} = \gamma^{212} + 2\gamma^{122}, \quad 0 = \gamma^2 \cdot \gamma^{21} = 2\gamma^{221} + \gamma^{212}$$

we need only compute $\mathbb{E}[\gamma^{221}]$.

We use a version of Green's theorem for the Young integral. Let η be the concatenation of γ with counter-clockwise arc from 1 to 0 along the boundary

of the disk, and $A(\gamma)$ be the region enclosed within this simple loop. In particular we wish to show that

$$\int_0^1 \eta_2^2(t) d\eta_1(t) = -2 \int_{A(\gamma)} y dx dy.$$

where the right integral should be understood in the Lebesgue sense.

Given a curve γ in $\mathcal{V}^p(\mathbb{R}^2)$ for some $p < 2$, by Proposition 4.2, it is possible to approximate it arbitrarily well in $\mathcal{V}^q(\mathbb{R}^2)$ and supremum norm for some $p < q < 2$ by piecewise linear curves, which may be assumed to each be simple by Lemma 4.3. Let γ_n be such a sequence of approximations to the SLE curve γ . Let η_n be the concatenation of this piecewise linear approximation with the counter-clockwise arc from 1 to 0 along the boundary of the disk. By the continuity properties of the Young integral, as stated in Theorem 4.1, we know that

$$\int_0^1 (\eta_n)_2^2(t) d(\eta_n)_1(t) \rightarrow \int_0^1 \eta_2^2(t) d\eta_1(t) \text{ as } n \rightarrow \infty.$$

As these are now piecewise smooth, we know by Green's theorem that

$$-2 \int_{A(\gamma_n)} y dx dy \rightarrow \int_0^1 \eta_2^2(t) d\eta_1(t) \text{ as } n \rightarrow \infty.$$

Thus, the argument is complete as long as

$$\int_{A(\gamma_n)} y dx dy \rightarrow \int_{A(\gamma)} y dx dy \text{ as } n \rightarrow \infty.$$

Since the γ_n tend towards γ in supremum norm, the desired convergence holds as long as the area within ε of the curve γ tends to zero as ε tends to zero. This is ensured by the almost sure order of Hölder continuity of γ , completing the proof of the required instance of Green's theorem.

One may see (by another similar approximation argument, or an application of the shuffle product $\gamma^2\gamma^2$ on the curve up to the time t) that

$$\begin{aligned} \gamma^{221} &= \int_0^1 \int_0^t \int_0^{t_2} d\gamma_2(t_1) d\gamma_2(t_2) d\gamma_1(t) \\ &= \frac{1}{2} \int_0^1 \gamma_2^2(t) d\gamma_1(t) \end{aligned}$$

and hence that

$$\gamma^{221} = \frac{1}{2} \left(\frac{1}{8} \int_0^\pi \sin^3(\theta) d\theta - 2 \int_{A(\gamma)} y dx dy \right) = \frac{1}{12} - \int_{A(\gamma)} y dx dy.$$

Thus, all that needs to be understood is

$$\mathbb{E} \left[\int_{A(\gamma)} y dx dy \right] = \int_D y \mathbb{P}\{x + iy \in A(\gamma)\} dx dy.$$

However, $\mathbb{P}\{x + iy \in A(\gamma)\} = p(x, y)$ and hence we have the first formula.

To obtain the more explicit formula, we need to work with the explicit definition of $p(x, y)$. Let $g(z) = iz/(1 - z)$ be the conformal map from D to \mathbb{H} used in the definition of $p(x, y)$ and $f(w) := w/(w+i)$ be its inverse. Examining the integral, and changing variables to \mathbb{H} by setting $z = x + iy = f(w)$

$$\begin{aligned} \int_D yp(x, y) dx dy &= \int_D \text{Im}(z)\phi(\arg(g(z))) dA(z) \\ &= \int_{\mathbb{H}} \text{Im}(f(w))\phi(\arg(w))|f'(w)|^2 dA(w) \\ &= \int_{\mathbb{H}} \text{Im}(w/(w+i))\phi(\arg(w))|w+i|^{-4} dA(w) \end{aligned}$$

Changing to polar coordinates yields

$$\begin{aligned} &\int_{\mathbb{H}} \text{Im}(f(w))\phi(\arg(w))|w+i|^{-4} dA(w) \\ &= - \int_0^\pi \phi(\theta) \cos(\theta) \int_0^\infty \frac{r^2}{(r^2 + 1 + 2r \sin \theta)^3} dr d\theta \\ &= \int_0^{\pi/2} (1 - 2\phi(\theta)) \cos(\theta) \int_0^\infty \frac{r^2}{(r^2 + 1 + 2r \sin \theta)^3} dr d\theta \end{aligned}$$

where the last line follows by the symmetries of \sin , \cos and ϕ .

A lengthy computation shows that the inner integral can be computed exactly. The result is

$$\begin{aligned} H(\theta) &:= \cos(\theta) \int_0^\infty \frac{r^2}{(r^2 + 1 + 2r \sin \theta)^3} dr \\ &= \frac{(2 \sin^2(\theta) + 1)(\frac{\pi}{2} - \theta - \sin(\theta) \cos(\theta))}{8 \cos^4(\theta)} - \frac{\tan(\theta)}{4}. \end{aligned}$$

It will be convenient to later reparametrize, so note that

$$H\left(\frac{\pi}{2} - \theta\right) = \frac{3\theta - 2\theta \sin^2(\theta) - 3 \cos(\theta) \sin(\theta)}{8 \sin^4(\theta)}.$$

By inserting the definition of $\phi(\theta)$, reorganizing, and applying Fubini's theorem

$$\begin{aligned} & \int_0^{\pi/2} (1 - 2\phi(\theta)) H(\theta) d\theta \\ &= 2C_\kappa \int_0^{\pi/2} H(\theta) \int_\theta^{\pi/2} \sin^\lambda(t) dt d\theta \\ &= 2C_\kappa \int_0^{\pi/2} H\left(\frac{\pi}{2} - \theta\right) \int_0^\theta \cos^\lambda(t) dt d\theta \\ &= 2C_\kappa \int_0^{\pi/2} \cos^\lambda(t) \int_t^{\pi/2} H\left(\frac{\pi}{2} - \theta\right) d\theta dt. \end{aligned}$$

One may again compute the inner integral exactly and obtain

$$\int_t^{\pi/2} H\left(\frac{\pi}{2} - \theta\right) d\theta = \frac{1}{8} \left(1 - \frac{\sin(t) - t \cos(t)}{\sin^3(t)} \right).$$

Substituting this back in to the integral in question, and rearranging yields

$$\int_D yp(x, y) dx dy = \frac{1}{8} - \frac{C_\kappa}{4} \int_0^{\pi/2} \frac{\sin(t) - t \cos(t)}{\sin^3(t)} \cos^\lambda(t) dt.$$

■

5. ANNULUS CROSSING PROBABILITIES

In this section we prove our regularity result by providing the bound on annulus crossing probabilities for SLE.

5.1. Notation and Topology. We let $B_r(z)$ denote the closed ball of radius r around z , and $C_r(z)$ denote the circle of radius r around z .

Let $A_r^R(z)$ denote the open annulus with inner radius r and outer radius R centered on z . Let $\gamma : [0, \infty] \rightarrow \mathbb{D}$ be a chordal SLE from 1 to -1 in the unit disk, considered under the standard capacity parametrization, and let D_t be the component of $\mathbb{D} \setminus \gamma[0, t]$ which contains -1 .

We wish to understand the probability that γ crosses $A_r^R(z)$ k times. Fixing an annulus $A_r^R(z)$, let $\mathcal{C}_k = \mathcal{C}_k(z; r, R)$ denote the set of simple curves from 1 to -1 that crosses the annulus precisely k times.

To be precise in our definition of crossing, we define the following set of recursive stopping times. In all the definitions, the infimums are understood to be infinity if taken over an empty set. Let $\tau_0 = \inf\{t > 0 \mid \gamma(t) \notin A_r^R(z)\}$. This is the first time that the SLE is not contained within the annulus. In the case that the annulus is bounded away from 1, this time is zero.

We now proceed recursively as follows. Assuming $\tau_i < \infty$, let L_i be the random variable taking values in the set $\{I, O\}$ where

$$L_i = \begin{cases} I & \gamma(\tau_i) \in B_r(z), \\ O & \gamma(\tau_i) \in \overline{\mathbb{D} \setminus B_R(z)}. \end{cases}$$

This random variable encodes the position of the curve at τ_i , taking the value I if it is inside the annulus, and O if it is outside.

Assuming $\tau_i < \infty$, define

$$\tau_{i+1} = \begin{cases} \inf\{t > \tau_i \mid \gamma(t) \in B_r(z)\} & L_i = O, \\ \inf\{t > \tau_i \mid \gamma(t) \in \overline{\mathbb{D} \setminus B_R(z)}\} & L_i = I. \end{cases}$$

In words, τ_{i+1} is the first time after τ_i that the curve γ completes a traversal from the inside of the annulus to the outside, or from the outside of the annulus to the inside. By continuity of γ , we know $\tau_{i+1} > \tau_i$. We call the times τ_i , for $i \geq 1$, *crossing times*. In this notation, \mathcal{C}_k is precisely the set of curves such that $\tau_k < \infty$ and $\tau_{k+1} = \infty$.

Let $\sigma_i = \sup\{t < \tau_i \mid \gamma(t) \notin A_r^R(z)\}$, which is to say the last entrance time of γ in to the annulus, before crossing. These are not stopping times, but it is useful to have them to aid in our definitions. We call the curve segments $\gamma_i := \gamma[\sigma_i, \tau_i]$ *crossing segments*. An illustration of the definitions so far are given in Figure 2.

When we wish to estimate the probability that SLE performs various crossings, we will need some way of telling which crossings will require a decrease in probability. For instance, in Figure 2, a crossing as between σ_8 and τ_8 cannot be of small probability given the curve up to the time τ_7 since the curve must leave the annulus in order to reach -1 . As we will see in Section 5.2, the right way to handle this is to keep track of the *crossing distance* of the tip to -1 , which we denote by Δ_t .

We define this notion as follows. Let E_t , the set of extensions of $\gamma[0, t]$, denote the set of simple curves $\eta : [0, \infty] \rightarrow \mathbb{D}$ so that η agrees with γ up to time t with $\eta(\infty) = -1$. We then define

$$\Delta_t = \min\{k \geq 0 \mid \mathcal{C}_k \cap E_t \neq \emptyset\} - \#\{k > 0 \mid \tau_k \leq t\},$$

which is to say the minimum number of crossings needed after time t to be consistent with the curve up to time t . As an example, the sequence of values of Δ_{τ_i} for $0 \leq i \leq 8$ from Figure 2 are $(0, 1, 0, 1, 2, 3, 2, 1, 0)$. By considering such examples, we quickly arrive at the following lemma.

Lemma 5.1. *Given the above definitions, the followings statements all hold when $-1 \notin A_r^R(z)$:*

- (1) Δ_t is integer valued and non-negative,

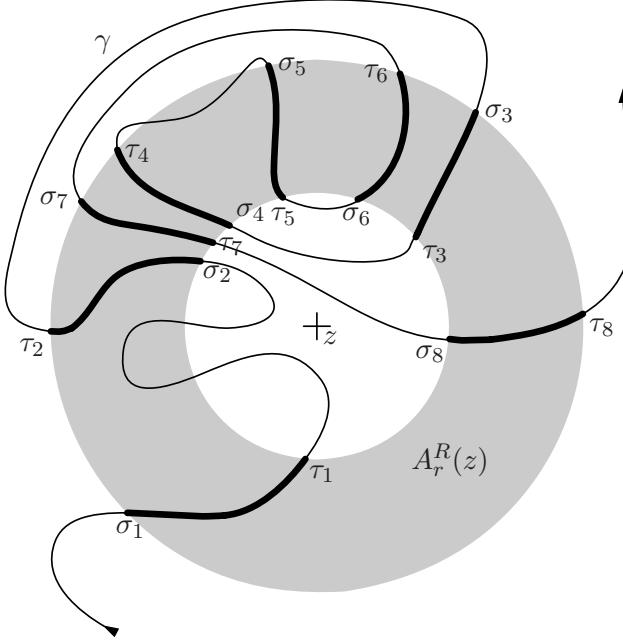


FIGURE 2. An illustration of the definitions given so far. This picture should be understood as all strictly contained within \mathbb{D} . All points along the curve γ are labeled by the time the curve crosses the point, not by the point itself. The crossing segments are indicated in bold.

- (2) given $t_1 < t_2$ so that there is no $i \geq 1$ with $t_1 < \tau_i \leq t_2$, we have $\Delta_{t_1} = \Delta_{t_2}$, and
- (3) $|\Delta_{\tau_{i+1}} - \Delta_{\tau_i}| = 1$.

Proof. The first statement is immediate from the definition. To prove the second, we proceed by showing that each side is an upper bound for the other.

First, $\#\{k > 0 \mid \tau_k \leq t_1\} = \#\{k > 0 \mid \tau_k \leq t_2\}$ since there is no τ_i with $t_1 < \tau_i \leq t_2$. Thus $\Delta_{t_2} \geq \Delta_{t_1}$ since every element of E_{t_2} is an element of E_{t_1} .

To see the opposite inequality, we will take an element of $E_{t_1} \cap \mathcal{C}_{\Delta_{t_1}}$ and produce an element of $E_{t_2} \cap \mathcal{C}_{\Delta_{t_1}}$, thus proving the opposite inequality. Let η be such a curve in $E_{t_1} \cap \mathcal{C}_{\Delta_{t_1}}$. Let $t^* = \max\{t \geq t_1 \mid \eta(t) \in \gamma[t_1, t_2]\}$, and let $t_* = \max\{t \leq t_2 \mid \gamma(t) = \eta(t^*)\}$. Both of these exist by compactness and continuity of γ and η . We construct η' as follows. First, follow the curve γ up to time t_2 , then follow the curve γ backwards from t_2 until time t_* , and then follow η from t^* until it reaches -1 .

The curve η' is not an element of E_{t_2} since it retraces its path in reverse between t_2 and t_* , however otherwise it is simple. By openness of D_{t_2} , we may perturb the curve so that after t_2 , rather than retrace γ exactly, it follows

a similar path in D_{t_2} which eventually continues as η , but still never crosses $A_r^R(z)$, yielding a curve η'' (see Figure 3 for an illustration of this process in an alternate case, which we will use later, where $t_1 = \tau_i$ and $t_2 = \tau_{i+1}$). This curve does not have any more crossings than η by construction, but also can have no fewer by our choice of η . thus η'' is the desired element in $E_{t_2} \cap \mathcal{C}_{\Delta_{t_1}}$.

We now prove the third item. First note that $\Delta_{\tau_{i+1}} \geq \Delta_{\tau_i} - 1$ since $E_{\tau_{i+1}}$ is contained in E_{τ_i} and $\#\{k > 0 \mid \tau_k \leq \tau_{i+1}\} = \#\{k > 0 \mid \tau_k \leq \tau_i\} + 1$.

By the same construction as above, we may take a curve $\eta \in E_{\tau_{i+1}}$ and produce a curve $\eta'' \in E_{\tau_i}$ with at most two more crossing of the annulus than η . Thus $\Delta_{\tau_{i+1}} \leq \Delta_{\tau_i} + 1$.

We can complete proof of the lemma as long as we can show that $\Delta_{\tau_{i+1}} \neq \Delta_{\tau_i}$. However, this follows immediately since $-1 \notin A_r^R(z)$ and hence the parity of Δ_{τ_i} must alternate (since we know which boundary of $A_r^R(z)$ must be passed through last and $\gamma(\tau_i)$ alternates which boundary it is contained in by definition). \blacksquare

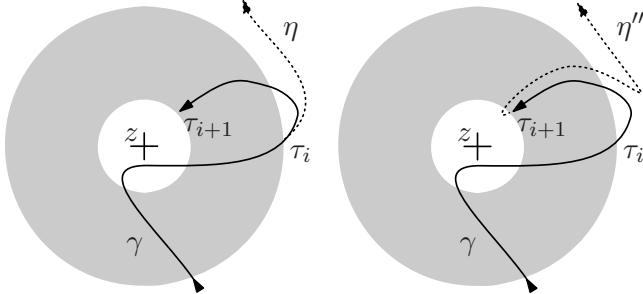


FIGURE 3. An example of the construction of η'' from η .

Note that everything besides the third bullet point in the above lemma would hold for any annulus, including those containing -1 . This issue will return later when our main estimate will need an extended proof when $-1 \in A_r^R(z)$.

As Δ_t is constant on away from the crossing times, we often suppress the exact dependence on time, and let $\Delta_i := \Delta_{\tau_i}$. As they will play a special role in the proof, we call the times τ_{i+1} such that $\Delta_{i+1} = \Delta_i + 1$ the *times of increase*, and the times such that $\Delta_{i+1} = \Delta_i - 1$ the *time of decrease*. We may further refine our understanding of the times of increase with the following lemma.

Lemma 5.2. *Fix an annulus $A_r^R(z)$ not containing -1 . Let ξ_1 be the arc of $D_{\tau_i} \cap (C_r(z) \cup C_R(z))$ which contains $\gamma(\sigma_{i+1})$, and ξ_2 be the arc of $D_{\tau_i} \cap (C_r(z) \cup C_R(z))$ which contains $\gamma(\tau_{i+1})$. Then, τ_{i+1} is a time of increase if and only if ξ_1 separates $\gamma(\tau_i)$ and -1 from ξ_2 in D_{τ_i} .*

Proof. First, if ξ_1 separates ξ_2 from -1 , every $\eta \in E_{\tau_{i+1}}$ intersects ξ_1 after τ_{i+1} . Let $t^* = \sup\{t \geq \tau_{i+1} \mid \gamma(t) \in \xi_1\}$. Consider the curve $\eta' \in E_{\tau_i}$ which is constructed by following η until σ_{i+1} , then following ξ_1 between the points $\gamma(\sigma_{i+1})$ and $\gamma(t^*)$ and then following η again after t^* . This curve has at least two fewer crossings of the annulus after the time τ_i than η did, and hence by taking η as a minimizer for the crossing distance, we see that $\Delta_{i+1} \geq \Delta_i + 1$ and hence τ_{i+1} is a time of increase.

Thus we need only show the converse. Note that ξ_1 always separates $\gamma(\tau_i)$ from ξ_2 in D_{τ_i} and thus we need only show ξ_1 separates -1 from ξ_2 in D_{τ_i} . We do so by showing that if ξ_2 is not separated from -1 in D_{τ_i} then τ_{i+1} must be a time of decrease. To do so we perform a construction very similar to the previous lemma. Take $\eta \in E_{\sigma_{i+1}}$ so that it minimizes $\Delta_{\sigma_{i+1}} = \Delta_i$. First, note that η may be assumed to be contained entirely in the component of $D_{\tau_i} \setminus \xi_1$ that contains both ξ_1 in its interior and -1 in its boundary after the time σ_{i+1} since otherwise we may follow very near to ξ_1 between σ_{i+1} and η 's last crossing of ξ_1 and obtain a new curve that stays within the desired component and certainly has no more crossings than η .

Now, we construct a curve $\eta' \in E_{\tau_{i+1}}$ with at most one more crossing of $A_r^R(z)$ than η . We take η' to be a simple curve very near to the curve formed by following γ until time τ_{i+1} and then following the reversal of γ back to time σ_{i+1} and then following η after time σ_{i+1} . By our choice of η to stay within $D_{\tau_i} \setminus \xi_1$ after σ_{i+1} , we may choose η' to have at most one more crossing of $A_r^R(z)$ than η (which is the crossing that occurred between σ_{i+1} and τ_{i+1}) and hence $\Delta_{i+1} \leq \Delta_i$ showing τ_{i+1} is a time of decrease. ■

5.2. Crossing bounds. With the above definitions, we may prove our main estimates.

First, we recall the definition of excursion measure, which is a conformally invariant notion of distance between boundary arcs in a simply connected domain (see, for example, [8]). Let D be a simply connected domain and let V_1, V_2 be two boundary arcs. As it is all we use, we assume the boundary arcs are C^1 . If it were needed, conformal invariance would allow us to extend this definition to arbitrary boundaries. Let $h_D(z)$ denote the probability that a brownian motion started at z exits D through V_2 . Then the *excursion measure* between V_1 and V_2 is defined to be

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \partial_{\mathbf{n}} h_D(z) |dz|$$

where $\partial_{\mathbf{n}}$ denotes the normal derivative.

Given a pair of disjoint simple C^1 curves $\xi_1, \xi_2 : (0, 1) \rightarrow D$ in D , then we write $\mathcal{E}_D(\xi_1, \xi_2)$ for the excursion measure between ξ_1 and ξ_2 in the unique component of $D \setminus (\xi_1(0, 1), \xi_2(0, 1))$ which has both ξ_1 and ξ_2 on the boundary.

To relate this to probabilities involving SLE, we need a lemma which can be found in [12, Lemma 4.5]. The statement here is slightly modified from the version there, but the proof follows immediately from an application of the monotonicity of excursion measure.

Lemma 5.3. *There exists a $c > 0$ so the following holds. Let D be a domain, and let γ be a chordal SLE_κ path from z_1 to z_2 in D . Let $\xi_1, \xi_2 : (0, 1) \rightarrow$ be a pair of curves so that $\xi_i(0^+)$ and $\xi_i(1^-)$ are both in ∂D so that ξ_1 separates ξ_2 from z_1 and z_2 . Then*

$$\mathbb{P}\{\gamma[0, \infty] \cap \xi_2(0, 1) \neq \emptyset\} \leq c \mathcal{E}_D(\xi_1, \xi_2)^\beta$$

where $\beta = 4a - 1$.

We apply this lemma when ξ_1 and ξ_2 are arcs contained in $\partial A_r^R(z)$, and hence we wish to bound the size of the excursion measure between two such arcs. We use the Beurling estimate (see, for example, [8, Theorem 3.76]) as the main tool in providing this bound. Given a Brownian motion B_t , let $\tau_{\mathbb{D}} = \inf\{t > 0 \mid B_t \notin \mathbb{D}\}$.

Theorem 5.4 (Beurling estimate). *There is a constant $c < \infty$ such that if $\gamma : [0, 1] \rightarrow \mathbb{C}$ is a curve with $\gamma(0) = 0$ and $|\gamma(1)| = 1$, $z \in \mathbb{D}$, and B_t is a Brownian motion, then*

$$\mathbb{P}^z\{B[0, \tau_{\mathbb{D}}] \cap \gamma[0, 1] = \emptyset\} \leq c|z|^{1/2}.$$

Combining this with a number of standard Brownian motion estimates (see, for example [19]), we may provide the following estimate.

Lemma 5.5. *There exists a $c < \infty$ so the following holds. Let $r < R/16$, $\gamma : (0, 1) \rightarrow A_r^R(z_0)$ be a curve with $\gamma(0^-) \in C_r(z_0)$ and $\gamma(1^+) \in C_R(z_0)$, and $U = A_r^R(z_0) \setminus \gamma(0, 1)$. Let ξ_1 be an open arc in $C_r(z_0)$ subtending an angle θ_1 and ξ_2 be an open arc in $C_R(z)$ subtending an angle θ_2 . Then*

$$\mathcal{E}_U(\xi_1, \xi_2) \leq c \theta_1 \theta_2 \left(\frac{r}{R}\right)^{1/2}.$$

Proof. We bound $h_U((r+\varepsilon)e^{i\theta})$ by splitting into three steps: the probability the Brownian motion reaches radius $2r$ (providing the bound needed to take the derivative), the probability it reaches radius $R/2$ (providing the dependence on r/R), and finally the probability it hits ξ_2 if it reaches radius R (providing dependence on θ_2). Integrating this bound over ξ_1 provides the desired bound on the excursion measure.

First, if the Brownian motion is to reach ξ_2 , it must reach $C_{2r}(z)$. By considering the gambler's ruin estimate applied to a Brownian motion motion started at $(r + \varepsilon)e^{i\theta}$ in the annulus $A_r^{2r}(z_0)$, the probability that the Brownian

motion reaches $C_{2r}(z_0)$ is at most

$$\frac{\log(r + \varepsilon) - \log(r)}{\log(2r) - \log(r)} = \log\left(1 + \frac{\varepsilon}{r}\right) \leq \frac{\varepsilon}{r}.$$

Second, to estimate the probability that the Brownian motion travels from $C_{2r}(z_0)$ to $C_{R/2}(z_0)$ avoiding γ , we apply the Beurling estimate (Theorem 5.4). This yields the bound of $c(r/R)^{1/2}$ by considering the curve in the annulus $A_{4r}^{R/2-2r}(\gamma(0^-))$ (which is non-degenerate, and has a ratio of radii comparable to r/R since we assumed $r < R/16$).

Finally, we wish to estimate the probability that a Brownian motion starting on $C_{R/2}(z_0)$ hits an arc subtending an angle of θ_2 located on $C_R(z_0)$. By an explicit computation with the Poisson kernel in \mathbb{D} , we obtain that this probability is bounded above by $c\theta_2$.

Using the strong Markov property, we may combine these estimates yielding

$$h_U((r + \varepsilon)e^{i\theta}) \leq c \varepsilon \frac{1}{r} \theta_2 \left(\frac{r}{R}\right)^{1/2}, \quad \partial_n h_U(re^{i\theta}) \leq c \frac{1}{r} \theta_2 \left(\frac{r}{R}\right)^{1/2}$$

and hence

$$\mathcal{E}_U(\xi_1, \xi_2) = \int_{\xi_1} \partial_n h_U(z) |dz| \leq c \theta_1 \theta_2 \left(\frac{r}{R}\right)^{1/2}.$$

■

The above lemma allows us to show the occurrence of a time of increase must be paid for with a corresponding cost in probability.

Proposition 5.6. *There exists a $c > 0$ so that for $i \geq 1$*

$$\mathbb{P}\{\tau_{i+1} < \infty ; \Delta_{i+1} = \Delta_i + 1 \mid \mathcal{F}_{\tau_i}\} \leq c \mathbf{1}\{\tau_i < \infty\} \left(\frac{r}{R}\right)^{\beta/2}.$$

Proof. If $\tau_i = \infty$, the curve cannot cross again, and hence we may restrict to the complementary case.

We wish to bound the probability that $\tau_{i+1} < \infty$ and it is a time of increase. Let ξ_1 and ξ_2 be a pair of arcs of $\partial A_r^R(z_0) \cap D_{\tau_i}$ which could contain σ_{i+1} and τ_{i+1} respectively (by which we mean a pair of arcs with ξ_1 on the same component of $\partial A_r^R(z_0)$ as τ_i and ξ_2 on the opposite boundary component such that both arcs are in the boundary of a single component $U \in S_{\tau_i}$).

Since $i \geq 1$ the annulus has been crossed at least once by time τ_i . Thus, ξ_1 and ξ_2 are contained in the boundary of some domain $U \in S_{\tau_i}$ which is bounded by some crossing segment γ_j . By monotonicity of excursion measure, we may use Lemma 5.5 to conclude that

$$\mathcal{E}_U(\xi_1, \xi_2) \leq \mathcal{E}_{A_r^R(z_0) \setminus \gamma_j}(\xi_1, \xi_2) \leq c \theta_1 \theta_2 \left(\frac{r}{R}\right)^{1/2}$$

where θ_1 and θ_2 are the angles subtended by the arcs ξ_1 and ξ_2 .

Since τ_{i+1} is time of increase, Lemma 5.2 implies that ξ_1 separates ξ_2 from both $\gamma(\tau_i)$ and -1 as needed for Lemma 5.3. Restricting to such arcs we see

$$\begin{aligned} \mathbb{P}\{\gamma[\tau_i, \infty] \cap \xi_2(0, 1) \neq \emptyset \mid \mathcal{F}_{\tau_i}\} &\leq c \mathcal{E}_{D_{\tau_i}}(\xi_1, \xi_2)^\beta \\ &\leq c \theta_1^\beta \theta_2^\beta \left(\frac{r}{R}\right)^{\beta/2} \\ &\leq c \theta_1 \theta_2 \left(\frac{r}{R}\right)^{\beta/2} \end{aligned}$$

where c is being used generically, and the $\beta = 4a - 1$ may be removed from the θ_i since $\theta_i \leq 2\pi$ and $\beta \geq 1$ when $\kappa \leq 4$.

We now conclude the bound by summing over all possible pairs of arcs satisfying the above criteria. We use the extremely weak bound that perhaps *every* pair of arcs, one on the interior boundary and one on the exterior boundary, might satisfy these conditions. By summing over all such pairs we see

$$\begin{aligned} \mathbb{P}\{\tau_{i+1} < \infty ; \Delta_{i+1} = \Delta_i + 1 \mid \mathcal{F}_{\tau_i}\} &\leq \sum_{\xi_1, \xi_2} \mathbb{P}\{\tau_{i+1} < \infty ; \Delta_{i+1} = \Delta_i + 1 ; \gamma(\sigma_{i+1}) \in \xi_1 ; \gamma(\tau_{i+1}) \in \xi_2 \mid \mathcal{F}_{\tau_i}\} \\ &\leq \sum_{\xi_1, \xi_2} \mathbb{P}\{\gamma[\tau_i, \infty] \cap \xi_2(0, 1) \neq \emptyset \mid \mathcal{F}_{\tau_i}\} \\ &\leq c \sum_{\xi_1, \xi_2} \theta_1 \theta_2 \left(\frac{r}{R}\right)^{\beta/2} \\ &= 4\pi^2 c \left(\frac{r}{R}\right)^{\beta/2}. \end{aligned}$$

■

The restriction that there is already at least one crossing is necessary in the above lemma. By using the SLE Green's function (see, for example, [11, 12]), the probability of at least a single crossing is of the order $(r/R)^{2-d}$ for an annulus contained in \mathbb{D} bounded away from -1 and 1 . This is a weaker bound for $\kappa < 4$ than the one obtained above. Additionally, an SLE must cross any annulus with 1 and -1 in separate components of $\mathbb{D} \setminus A_r^R(z_0)$ at least once. Since we need the bound to hold uniformly for all annuli but we do not need the exponents to be optimal, we use the trivial bound of 1 for the first crossing. More care must be taken in this estimate if a sharper exponent is desired.

We now obtain our bound on the number of annulus crossings by simple combinatorial estimates enforced by the form of Δ_i as a function of i given in Lemma 5.1. These paths are closely related to *Dyck paths*. A Dyck path of length k is a walk on \mathbb{N} with $2k$ steps of ± 1 , which both starts and ends at zero. Let C_k denote the total number of Dyck paths of length k . Since the

path starts and ends at zero, there must be the same number of $+1$ steps as -1 steps.

Theorem 5.7. *There exist c_1, c_2 such that for any $k \geq 1$, $z_0 \in \mathbb{D}$, and $r < R$,*

$$\mathbb{P}\{\mathcal{C}_k(z_0; r, R)\} \leq c_1 \left(c_2 \frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)}.$$

Proof. Due to the topology of the situation, we must split this proof into two main cases: the case where -1 is not contained in $A_r^R(z_0)$ and the case where it is.

First, we prove the case $-1 \notin A_r^R(z_0)$ as the second case reuses much of the same argument. We proceed by splitting the event into those crossings which share a common sequence of values for Δ_i , bounding the probability of a particular sequence by repeated application of Proposition 5.6, and then relating the number of such sequences to the number of Dyck Paths to obtain the constant.

Take some curve in $\mathcal{C}_k(z_0; r, R)$, and consider the associated Δ_i for $0 \leq i \leq k$. In this case, $\Delta_k = 0$ since the curve proceeds to -1 without any further crossings. Also, depending on if $\gamma(\tau_0)$ and -1 are in the same component of $\mathbb{D} \setminus A_r^R(z_0)$ or not, we have that $\Delta_0 \in \{0, 1\}$ (this observation strongly uses that \mathbb{D} is convex and hence can force at most one crossing of the annulus). If $\Delta_0 = 1$, $\Delta_1 = 0$ since $\gamma(\tau_1)$ would have to be the first time γ was contained in the boundary of the component of $\mathbb{D} \setminus A_r^R(z_0)$ which contains -1 . Thus by Lemma 5.1, either $(\Delta_0, \dots, \Delta_k)$ or $(\Delta_1, \dots, \Delta_k)$ is a Dyck path of length $\lfloor k/2 \rfloor$ (and indeed since the number of steps in a Dyck path must be even, at most one of these cases can hold for any given k). In either case the Dyck path must contain exactly $\lfloor k/2 \rfloor$ steps of $+1$, which is to say at least $\lfloor k/2 \rfloor$ times of increase.

Thus, for a fixed Dyck path $\mathbf{d} = (d_0, \dots, d_{\lfloor k/2 \rfloor})$, we let $\mathcal{C}_{\mathbf{d}} \subseteq \mathcal{C}_k$ denote the set of curves for which either $(\Delta_0, \dots, \Delta_k) = \mathbf{d}$ or $(\Delta_1, \dots, \Delta_k) = \mathbf{d}$. On this event, after discarding the first time of increase, and applying Proposition 5.6 to each subsequent time of increase we obtain

$$\mathbb{P}\{\mathcal{C}_{\mathbf{d}}\} \leq c^{\lfloor k/2 \rfloor - 1} \left(\frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)}.$$

Finally, any such Dyck path \mathbf{d} of length $\lfloor k/2 \rfloor$ can occur, and hence we need to sum over all of the possibilities yielding

$$\begin{aligned} \mathbb{P}\{\mathcal{C}_k(z_0; r, R)\} &= \sum_{\mathbf{d}} \mathbb{P}\{\mathcal{C}_{\mathbf{d}}\} \\ &\leq c^{\lfloor k/2 \rfloor - 1} C_{\lfloor k/2 \rfloor} \left(\frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)}. \end{aligned}$$

Since $C_\ell \leq 4^\ell$, there are universal c_1 and c_2 so that

$$\mathbb{P}\{\mathcal{C}_k(z_0; r, R)\} \leq c_1 \left(\frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)}$$

as needed for the first case.

In the second case, where $-1 \in A_r^R(z_0)$, we may no longer apply the above reasoning as Lemma 5.1 no longer holds for our annulus. In particular, crossing the annulus need not change the number of crossings needed to reach -1 and hence we have no lower bound on the number of times of increase. Let $r_0 = |z_0 + 1|$ be the distance between the center of the annulus and -1 . To extend the proof to this case, we split the annulus into the pair of annuli $A_I := A_r^{r_0}(z_0)$ and $A_O := A_{r_0}^R(z_0)$ and produce an upper bound of the exact same order by running the same argument in parallel for both annuli.

Take any curve that crosses $A_r^R(z_0)$ exactly k times. Any such curve must cross both A_I and A_O at least k times. Let k_I be the number of times the curve crosses A_I and similarly let k_O be the number of times the curve crosses A_O . Let Δ_i^I and Δ_i^O be the associated crossing functions. As -1 is contained in neither A_I nor A_O , Lemma 5.1 applies and, as above, we may associate a Dyck path of length $\lfloor k_I/2 \rfloor$ to Δ_i^I and a Dyck path of length $\lfloor k_O/2 \rfloor$ to Δ_i^O .

We have no upper bound on k_O and k_I in comparison to k , and thus attempting to mirror the exact proof from above will not succeed since summing over all pairs of Dyck path of length at least $\lfloor k/2 \rfloor$ could yield a divergent sum since we have no control on the ratios r/r_0 and r_0/R . Thus we must make this sum finite depending only on k . Given a Dyck path of length $\ell' > \ell$ we must have at least ℓ times of increase in the first 2ℓ steps of that Dyck path as otherwise the path would be negative at step 2ℓ . Thus, as we only need $\lfloor k/2 \rfloor$ times of increase to obtain the desired bound, we need only consider the initial segments of the Dyck paths associated to Δ_i^I and Δ_i^O .

Thus, for two initial segments \mathbf{d}_I and \mathbf{d}_O of Dyck paths both containing exactly $2\lfloor k/2 \rfloor$ steps, we let $\mathcal{C}_{\mathbf{d}_I, \mathbf{d}_O} \subseteq \mathcal{C}_k$ denote the set of curves for which both either $(\Delta_0^I, \dots, \Delta_k^I) = \mathbf{d}_I$ or $(\Delta_1^I, \dots, \Delta_k^I) = \mathbf{d}_I$ and either $(\Delta_0^O, \dots, \Delta_k^O) = \mathbf{d}_O$ or $(\Delta_1^O, \dots, \Delta_k^O) = \mathbf{d}_O$. On this event, after discarding the first time of increase for both annuli, and applying Proposition 5.6 to the next $\lfloor k/2 \rfloor - 1$ subsequent time of increase we obtain

$$\begin{aligned} \mathbb{P}\{\mathcal{C}_{\mathbf{d}_I, \mathbf{d}_O}\} &\leq c^{2\lfloor k/2 \rfloor - 2} \left(\frac{r}{r_0}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)} \left(\frac{r_0}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)} \\ &= c^{2\lfloor k/2 \rfloor - 2} \left(\frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k/2 \rfloor - 1)}. \end{aligned}$$

Finally, noting that there are at most $16^{\lfloor k/2 \rfloor}$ different pairs of $\mathbf{d}_I, \mathbf{d}_O$, we may sum over each of these possibilities and obtain the desired bound in the same manner as the first case. \blacksquare

To apply the results of Aizenman and Burchard, we need to have a bound on the probability of having at least k_0 crossings. Such bound is easily obtained from the above by summing over all $k \geq k_0$ to see that

$$\mathbb{P}\{\gamma \text{ traverses } A_r^R(z_0) \text{ at least } k \text{ separate times}\} \leq c_{k_0} \left(\frac{r}{R}\right)^{\frac{\beta}{2}(\lfloor k_0/2 \rfloor - 1)}$$

where c_{k_0} is some new constant depending only on k_0 , thus completing the proof of the Theorem 1.2, and hence of all results.

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